

# Metrical Multi-Time Lagrange Geometry of Physical Fields

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## Abstract

Section 1 contains some physical and geometrical aspects of the Lagrangian geometry of physical fields developed by Miron and Anastasiei [7], which represents the start point in our metrical multi-time Lagrangian approach of the theory of physical fields. Section 2 exposes a geometrization of a Kronecker  $h$ -regular Lagrangian function with partial derivatives of order one,  $L : J^1(T, M) \rightarrow R$ . This geometrization relies on the notion of metrical multi-time Lagrange space  $ML_p^n = (J^1(T, M), L)$  introduced in [12]. We emphasize that this geometry gives a model for both the gravitational and electromagnetic field theory, in a general setting. Thus, Section 3 presents the metrical multi-time Lagrange theory of electromagnetism and describes its Maxwell equations. Section 4 presents the Einstein equations which govern the metrical multi-time Lagrange theory of gravitational field. The conservation laws of the gravitational field are also described in terms of metrical multi-time Lagrange geometry.

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## 1 Lagrangian theory of physical fields

A lot of geometrical models in Mechanics, Physics or Biology are based on the notion of ordinary Lagrangian. In this sense, we recall that a Lagrange space  $L^n = (M, L(x, y))$  is defined as a pair which consists of a real, smooth,  $n$ -dimensional manifold  $M$  coordinated by  $(x^i)_{i=\overline{1, n}}$ , and a regular Lagrangian  $L : TM \rightarrow R$ , not necessarily homogenous with respect to the direction  $(y^i)_{i=\overline{1, n}}$ . The differential geometry of Lagrange spaces is now considerably developed and used in various fields to study natural process where the dependence on position, velocity or momentum is involved [7]. Also, the geometry of Lagrange spaces gives a model for both the gravitational and electromagnetic field, in a very natural blending of the geometrical structure of the space with the characteristic properties of the physical fields.

In the sequel, we try to expose the main geometrical and physical aspects of *the Lagrangian theory of physical fields* [7]. In order to do that, let us consider

$$(1.1) \quad g_{ij}(x^k, y^k) = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j},$$

the *fundamental metrical d-tensor* of an ordinary Lagrangian  $L : TM \rightarrow R$ . From physical point of view, this d-tensor has the physical meaning of an "unified" gravitational field on  $TM$ , which consists of one "external" ( $x$ )-gravitational field spanned by points  $\{x\}$ , and the other "internal" ( $y$ )-gravitational field spanned by directions  $\{y\}$ . It should be emphasized that  $y$  is endowed with some microscopic character of the space-time structure. Moreover, since  $y$  is a vector field different of an ordinary vector field, the  $y$ -dependence has combined with the concept of *anisotropy*.

The field theory developed on a Lagrange space  $L^n$  relies on a nonlinear connection  $\Gamma = (N_j^i(x, y))$  attached naturally to the given Lagrangian  $L$ . This plays the role of mapping operator of internal ( $y$ )-field on the external ( $x$ )-field, and prescribes the "interaction" between ( $x$ )- and ( $y$ )- fields. From geometrical point of view, the nonlinear connection allows the construction of the *adapted bases*  $\left\{ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_j^i \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i} \right\} \subset \mathcal{X}(TM)$  and  $\{dx^i, \delta y^i = dy^i + N_j^i dx^j\} \subset \mathcal{X}^*(TM)$ .

Concerning the "unified" field  $g_{ij}(x, y)$  of  $L^n$ , the authors constructed a Sasakian-like metric on  $TM$ ,

$$(1.2) \quad G = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j.$$

As to the spatial structure, the most important thing is to determine the *Cartan canonical connection*  $CT = (L_{jk}^i, C_{jk}^i)$  with respect to  $g_{ij}$ , which comes from the metrical conditions

$$(1.3) \quad \begin{cases} g_{ij|k} = \frac{\delta g_{ij}}{\delta x^k} - L_{ik}^m g_{mj} - L_{jk}^m g_{mi} = 0 \\ g_{ij|k} = \frac{\partial g_{ij}}{\partial y^k} - C_{ik}^m g_{mj} - C_{jk}^m g_{mi} = 0, \end{cases}$$

where " $|_k$ " and " $|_k$ " are the local  $h$ - and  $v$ - covariant derivatives of  $CT$ . The importance to the Cartan canonical connection comes from its main role played in the Lagrangian theory of physical fields.

In this context, the Einstein equations of the gravitational potentials  $g_{ij}(x, y)$  of a Lagrange space  $L^n$ ,  $n > 2$ , are postulated as being the Einstein equations attached to  $CT$  and  $G$ , namely [7],

$$(1.4) \quad \begin{cases} R_{ij} - \frac{1}{2} R g_{ij} = \mathcal{K} \mathcal{T}_{ij}^H, & {}'P_{ij} = \mathcal{K} \mathcal{T}_{ij}^1, \\ S_{ij} - \frac{1}{2} S g_{ij} = \mathcal{K} \mathcal{T}_{ij}^V, & {}''P_{ij} = -\mathcal{K} \mathcal{T}_{ij}^2, \end{cases}$$

where  $R_{ij} = R_{ijm}^m$ ,  $S_{ij} = S_{ijm}^m$ ,  $'P_{ij} = P_{ijm}^m$ ,  $''P_{ij} = P_{imj}^m$  are the Ricci tensors of  $CT$ ,  $R = g^{ij} R_{ij}$ ,  $S = g^{ij} S_{ij}$  are the scalar curvatures,  $\mathcal{T}_{ij}^H$ ,  $\mathcal{T}_{ij}^V$ ,  $\mathcal{T}_{ij}^1$ ,  $\mathcal{T}_{ij}^2$  are the components of the energy-momentum tensor  $\mathcal{T}$  and  $\mathcal{K}$  is the Einstein constant (equal to 0 for vacuum). Moreover, the energy-momentum tensors  $\mathcal{T}_{ij}^H$  and  $\mathcal{T}_{ij}^V$  satisfy the following *conservation laws*

$$(1.5) \quad \mathcal{K} \mathcal{T}_{j|m}^H{}^m = -\frac{1}{2} (P_{js}^{hm} R_{hm}^s + 2 R_{mj}^s P_s^m), \quad \mathcal{K} \mathcal{T}_{j|m}^V{}^m = 0,$$

where all notations are described in [7].

The Lagrangian theory of electromagnetism relies on the *canonical Liouville vector field*  $\mathbf{C} = y^i \frac{\partial}{\partial y^i}$  and the Cartan canonical connection  $CT$  of the Lagrange space  $L^n$ . In this context, the authors introduce the *electromagnetic 2-form* on  $TM$ ,

$$(1.6) \quad F = F_{ij} \delta y^i \wedge dx^j + f_{ij} \delta y^i \wedge \delta y^j,$$

where

$$(1.7) \quad \begin{aligned} F_{ij} &= \frac{1}{2} [(g_{im} y^m)_{|j} - (g_{jm} y^m)_{|i}], \\ f_{ij} &= \frac{1}{2} [(g_{im} y^m)_{|j} - (g_{jm} y^m)_{|i}]. \end{aligned}$$

Using geometrical identities, they deduce that the vertical electromagnetic components  $f_{ij}$  vanish always.

At the same time, using the Bianchi identities attached to the Cartan canonical connection  $CT$ , they conclude that the horizontal electromagnetic components  $F_{ij}$  are governed by the following *equations of Maxwell type*,

$$(1.8) \quad \begin{cases} F_{ij|k} + F_{jk|i} + F_{ki|j} = - \sum_{\{i,j,k\}} C_{imr} R_{jk}^r y^m \\ F_{ij|k} + F_{jk|i} + F_{ki|j} = 0. \end{cases}$$

Finally, we point out that physical aspects of the Lagrangian electromagnetism are studied by Ikeda in [5].

In this paper, we naturally extend the previous field theory to a general one, constructed on the jet fibre bundle of order one  $J^1(T, M) \rightarrow T \times M$ , where  $T$  is a smooth, real,  $p$ -dimensional "*multi-time*" manifold coordinated by  $(t^\alpha)_{\alpha=1,p}$  and  $M$  is a smooth, real  $n$ -dimensional "*spatial*" manifold coordinated by  $(x^i)_{i=1,n}$ . The gauge group of  $J^1(T, M)$  is

$$(1.9) \quad \begin{cases} \tilde{t}^\alpha = \tilde{t}^\alpha(t^\beta) \\ \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{x}_\alpha^i = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{\partial t^\beta}{\partial \tilde{t}^\alpha} x_\beta^j. \end{cases}$$

In other words, it is more general than that used in the papers [7], [8]. We recall that the jet fibre bundle of order one is a basic object in the study of classical and quantum field theories.

Our field theory is created, in a natural manner, from a given *Kronecker  $h$ -regular* Lagrangian function on  $J^1(T, M)$  (i. e. a smooth function  $L : J^1(T, M) \rightarrow R$ ), and can be called the *metrical multi-time Lagrange theory of physical fields*.

In order to have a clear exposition of our theory, we point out that we use the following three distinct notions:

- i) *multi-time Lagrangian function* – A smooth function  $L : J^1(T, M) \rightarrow R$ .
- ii) *multi-time Lagrangian* (Olver's terminology) – A local function  $\mathcal{L}$  on  $J^1(T, M)$  which transform by the rule  $\tilde{\mathcal{L}} = \mathcal{L} |\det J|$ , where  $J$  is the Jacobian matrix of coordinate transformations  $t^\alpha = t^\alpha(\tilde{t}^\beta)$ . If  $L$  is a Lagrangian function on 1-jet fibre bundle, then  $\mathcal{L} = L \sqrt{|h|}$  represent a Lagrangian on  $J^1(T, M)$ .

iii) *multi-time Lagrangian density* (Marsden's terminology) – A smooth map  $\mathcal{D} : J^1(T, M) \rightarrow \Lambda^p(T^*T)$ . For example, the entity  $\mathcal{D} = \mathcal{L}dt^1 \wedge dt^2 \wedge \dots \wedge dt^p$ , where  $\mathcal{L}$  is a Lagrangian, represents a Lagrangian density on  $J^1(T, M)$ .

We emphasize that the construction of a theory of physical fields attached to a given first order multi-time Lagrangian function was tried unsatisfactory, again, by Miron, Kirkovits and Anastasiei in [8]. In their opinion, a such construction must be done on the vector bundle  $\oplus_1^p TM \rightarrow M$ , where the coordinates of  $\alpha$ -th copy of  $TM$  are denoted  $(x^i, x_\alpha^i)$ , and its gauge group is of the form

$$(1.10) \quad \begin{cases} \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{x}_\alpha^i = \frac{\partial \tilde{x}^i}{\partial x^j} x_\alpha^j. \end{cases}$$

In other words, their gauge group ignores the multi-temporal reparametrizations. From our point of view this is the first difficulty of their theory. At the same time, their trial was unsatisfactory because they do not succeeded to write the local expressions of the Bianchi identities of the Cartan canonical connection. This second difficulty of their theory appeared probably from the very complicated computations that was involved.

In our paper, we remove these difficulties, using a Kronecker  $h$ -regular multi-time Lagrange function on the 1-jet fibre bundle  $J^1(T, M)$ . These objects allow us the writing of the Bianchi identities of the Cartan canonical connection, so necessary in the description of field equations. Nevertheless, our theory has also a difficulty, coming from the quite strong condition of Kronecker  $h$ -regularity imposed to the multi-time Lagrangian function. This difficulty will be removed in the paper [9].

## 2 Metrical multi-time Lagrange spaces

Let us consider  $T$  (resp.  $M$ ) a "temporal" (resp. "spatial") manifold of dimension  $p$  (resp.  $n$ ), coordinated by  $(t^\alpha)_{\alpha=\overline{1,p}}$  (resp.  $(x^i)_{i=\overline{1,n}}$ ). Let  $E = J^1(T, M) \rightarrow T \times M$  be the jet fibre bundle of order one associated to these manifolds. The *bundle of configuration*  $J^1(T, M)$  is coordinated by  $(t^\alpha, x^i, x_\alpha^i)$ , where  $\alpha = \overline{1,p}$  and  $i = \overline{1,n}$ . Note that the terminology used above is justified in [11].

**Remarks 2.1** i) Throughout this paper, the indices  $\alpha, \beta, \gamma, \dots$  run from 1 to  $p$ , and the indices  $i, j, k, \dots$  run from 1 to  $n$ .

ii) In the particular case  $T = R$  (i. e., the temporal manifold  $T$  is the usual time axis represented by the set of real numbers), the coordinates  $(t^1, x^i, x_1^i)$  of the 1-jet space  $J^1(R, M) \equiv R \times TM$  are denoted  $(t, x^i, y^i)$ .

We start our study considering a smooth multi-time Lagrangian function  $L : E \rightarrow R$ , which is locally expressed by  $E \ni (t^\alpha, x^i, x_\alpha^i) \rightarrow L(t^\alpha, x^i, x_\alpha^i) \in R$ . The *vertical fundamental metrical d-tensor* of  $L$  is

$$(2.1) \quad G_{(i)(j)}^{(\alpha)(\beta)} = \frac{1}{2} \frac{\partial^2 L}{\partial x_\alpha^i \partial x_\beta^j}.$$

Now, let  $h = (h_{\alpha\beta})$  be a fixed semi-Riemannian metric on the temporal manifold  $T$  and  $g_{ij}(t^\gamma, x^k, x_\gamma^k)$  be a d-tensor on  $E$ , symmetric, of rank  $n$ , and having a constant

signature.

**Definition 2.1** A multi-time Lagrangian function  $L : E \rightarrow R$  whose vertical fundamental metrical d-tensor is of the form

$$(2.2) \quad G_{(i)(j)}^{(\alpha)(\beta)}(t^\gamma, x^k, x_\gamma^k) = h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma, x^k, x_\gamma^k),$$

is called a *Kronecker  $h$ -regular multi-time Lagrangian function with respect to the temporal semi-Riemannian metric  $h = (h_{\alpha\beta})$* .

In this context, we can introduce the following

**Definition 2.2** A pair  $ML_p^n = (J^1(T, M), L)$ , where  $p = \dim T$  and  $n = \dim M$ , which consists of the 1-jet fibre bundle and a Kronecker  $h$ -regular multi-time Lagrangian function  $L : J^1(T, M) \rightarrow R$  is called a *metrical multi-time Lagrange space*.

**Remarks 2.2** i) In the particular case  $(T, h) = (R, \delta)$ , a metrical multi-time Lagrange space is called a *relativistic rheonomic Lagrange space* and is denoted  $RL^n = (J^1(R, M), L)$ .

ii) If the temporal manifold  $T$  is 1-dimensional, then, via a temporal reparametrization, we have  $J^1(T, M) \equiv J^1(R, M)$ . In other words, a metrical multi-time Lagrangian space having  $\dim T = 1$  is a *reparametrized relativistic rheonomic Lagrange space*.

**Examples 2.1** i) Suppose that the spatial manifold  $M$  is also endowed with a semi-Riemannian metric  $g = (g_{ij}(x))$ . Then, the multi-time Lagrangian function

$$(2.3) \quad L_1 : J^1(T, M) \rightarrow R, \quad L_1 = h^{\alpha\beta}(t)g_{ij}(x)x_\alpha^i x_\beta^j$$

is a Kronecker  $h$ -regular multi-time Lagrangian function. Consequently,  $ML_p^n = (J^1(T, M), L_1)$  is a metrical multi-time Lagrange space. We underline that the multi-time Lagrangian  $\mathcal{L}_1 = L_1 \sqrt{|h|}$  is exactly the energy multi-time Lagrangian whose extremals are the harmonic maps between the pseudo-Riemannian manifolds  $(T, h)$  and  $(M, g)$  [3]. At the same time, this multi-time Lagrangian is a basic object in the physical theory of bosonic strings.

ii) In above notations, taking  $U_{(i)}^{(\alpha)}(t, x)$  as a d-tensor field on  $E$  and  $F : T \times M \rightarrow R$  a smooth map, the more general multi-time Lagrangian function

$$(2.4) \quad L_2 : E \rightarrow R, \quad L_2 = h^{\alpha\beta}(t)g_{ij}(x)x_\alpha^i x_\beta^j + U_{(i)}^{(\alpha)}(t, x)x_\alpha^i + F(t, x)$$

is also a Kronecker  $h$ -regular multi-time Lagrangian. The metrical multi-time Lagrange space  $ML_p^n = (J^1(T, M), L_2)$  is called the *autonomous metrical multi-time Lagrange space of electrodynamics* because, in the particular case  $(T, h) = (R, \delta)$ , we recover the classical Lagrangian space of electrodynamics [7] which governs the movement law of a particle placed concomitantly into a gravitational field and an electromagnetic one. From physical point of view, the semi-Riemannian metric  $h_{\alpha\beta}(t)$  (resp.  $g_{ij}(x)$ ) represents the *gravitational potentials* of the space  $T$  (resp.  $M$ ), the d-tensor  $U_{(i)}^{(\alpha)}(t, x)$  stands for the *electromagnetic potentials* and  $F$  is a function which is called *potential function*. The non-dynamical character of spatial gravitational potentials  $g_{ij}(x)$  motivates us to use the term "*autonomous*".

iii) More general, if we consider  $g_{ij}(t, x)$  a  $d$ -tensor field on  $E$ , symmetric, of rank  $n$  and having constant signature on  $E$ , we can define the Kronecker  $h$ -regular multi-time Lagrangian function

$$(2.5) \quad L_3 : E \rightarrow R, \quad L_3 = h^{\alpha\beta}(t)g_{ij}(t, x)x_\alpha^i x_\beta^j + U_{(i)}^{(\alpha)}(t, x)x_\alpha^i + F(t, x).$$

The pair  $ML_p^n = (J^1(T, M), L_3)$  is a metrical multi-time Lagrange space which is called the *non-autonomous metrical multi-time Lagrange space of electrodynamics*. Physically, we remark that the gravitational potentials  $g_{ij}(t, x)$  of the spatial manifold  $M$  are dependent of the temporal coordinates  $t^\gamma$ , emphasizing their dynamic character.

An important role and, at the same time, an obstruction in the subsequent development of the metrical multi-time Lagrangian geometry, is played by the next [11]

**Theorem 2.1** (*characterization of metrical multi-time Lagrange spaces*)

If we have  $\dim T \geq 2$ , then the following statements are equivalent:

- i)  $L$  is a Kronecker  $h$ -regular multi-time Lagrangian function on  $J^1(T, M)$ .
- ii) The multi-time Lagrangian function  $L$  reduces to a non-autonomous electrodynamics multi-time Lagrangian function, that is,

$$L = h^{\alpha\beta}(t)g_{ij}(t, x)x_\alpha^i x_\beta^j + U_{(i)}^{(\alpha)}(t, x)x_\alpha^i + F(t, x).$$

A direct consequence of the previous characterization theorem is the following

**Corollary 2.2** *The fundamental vertical metrical  $d$ -tensor of an arbitrary Kronecker  $h$ -regular multi-time Lagrangian function  $L$  is of the form*

$$(2.6) \quad G_{(i)(j)}^{(\alpha)(\beta)} = \frac{1}{2} \frac{\partial^2 L}{\partial x_\alpha^i \partial x_\beta^j} = \begin{cases} h^{11}(t)g_{ij}(t, x^k, y^k), & p = 1 \\ h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma, x^k), & p \geq 2, \end{cases}$$

where  $p = \dim T$ .

**Remarks 2.3** i) It is obvious that the preceding theorem is an obstruction in the development of a fertile metrical multi-time Lagrangian geometry. This obstruction will be removed in a subsequent paper by the introduction of a more general notion, that of *generalized metrical multi-time Lagrange space* [9]. The generalized metrical multi-time Lagrange geometry and its theory of physical fields are constructed in [9] using just a given  $h$ -regular fundamental vertical metrical  $d$ -tensor  $G_{(i)(j)}^{(\alpha)(\beta)}$  on the 1-jet space  $J^1(T, M)$ .

ii) In the case  $p = \dim T \geq 2$ , the above theorem obliges us to continue the study of the metrical multi-time Lagrangian space theory, channeling our attention upon the non-autonomous metrical multi-time Lagrange space of electrodynamics.

Following the geometrical development from the paper [12], the fundamental vertical metrical  $d$ -tensor  $G_{(i)(j)}^{(\alpha)(\beta)}$  of the metrical multi-time Lagrange space  $ML_p^n = (J^1(T, M), L)$  induces naturally a *canonical nonlinear connection*  $\Gamma = (M_{(\alpha)\beta}^{(i)}, N_{(\alpha)j}^{(i)})$  on  $E = J^1(T, M)$ .

**Theorem 2.3** *The canonical nonlinear connection  $\Gamma$  of the metrical multi-time Lagrange space  $ML_p^n = (J^1(T, M), L)$  is defined by the temporal components*

$$(2.7) \quad M_{(\alpha)\beta}^{(i)} = \begin{cases} -H_{11}^1 y^i, & p = 1 \\ -H_{\alpha\beta}^\gamma x_\gamma^i, & p \geq 2 \end{cases}$$

and the spatial components

$$(2.8) \quad N_{(\alpha)j}^{(i)} = \begin{cases} h_{11} \frac{\partial \mathcal{G}^i}{\partial y^j}, & p = 1 \\ \Gamma_{jk}^i x_\alpha^k + \frac{g^{ik}}{2} \frac{\partial g_{jk}}{\partial t^\alpha} + \frac{g^{ik}}{4} h_{\alpha\beta} U_{(k)j}^{(\beta)}, & p \geq 2, \end{cases}$$

where

$$(2.9) \quad \begin{aligned} \mathcal{G}^i &= \frac{g^{ik}}{4} \left( \frac{\partial^2 L}{\partial x^j \partial y^k} y^j - \frac{\partial L}{\partial x^k} + \frac{\partial^2 L}{\partial t \partial y^k} + \frac{\partial L}{\partial x^k} H_{11}^1 + 2h^{11} H_{11}^1 g_{kl} y^l \right), \\ H_{\alpha\beta}^\gamma &= \frac{h^{\gamma\eta}}{2} \left( \frac{\partial h_{\eta\alpha}}{\partial t^\beta} + \frac{\partial h_{\eta\beta}}{\partial t^\alpha} - \frac{\partial h_{\alpha\beta}}{\partial t^\eta} \right), \\ \Gamma_{jk}^i &= \frac{g^{im}}{2} \left( \frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \right), \\ U_{(k)j}^{(\beta)} &= \frac{\partial U_{(k)}^{(\beta)}}{\partial x^j} - \frac{\partial U_{(j)}^{(\beta)}}{\partial x^k}. \end{aligned}$$

**Remarks 2.4** i) Considering the particular case  $(T, h) = (R, \delta)$ , we remark that the canonical nonlinear connection  $\Gamma = (0, N_{(1)j}^{(i)})$  of the relativistic rheonomic Lagrange space  $RL^n = (J^1(R, M), L)$  reduces to the canonical nonlinear connection from Miron-Anastasiei theory [7].

ii) In the case of an autonomous metrical multi-time Lagrange space of electrodynamics (i. e.,  $g_{ij}(t^\gamma, x^k, x_\gamma^k) = g_{ij}(x^k)$ ), the generalized Christoffel symbols  $\Gamma_{jk}^i(t^\mu, x^m)$  of the metrical d-tensor  $g_{ij}$  reduce to the classical ones  $\gamma_{jk}^i(x^m)$ , and the canonical nonlinear connection becomes  $\Gamma = (M_{(\alpha)\beta}^{(i)}, N_{(\alpha)j}^{(i)})$ , where

$$M_{(\alpha)\beta}^{(i)} = \begin{cases} -H_{11}^1 y^i, & p = 1 \\ -H_{\alpha\beta}^\gamma x_\gamma^i, & p \geq 2 \end{cases} \quad \text{and} \quad N_{(\alpha)j}^{(i)} = \begin{cases} \gamma_{jk}^i y^k + \frac{g^{ik}}{4} h_{11} U_{(k)j}^{(1)}, & p = 1 \\ \gamma_{jk}^i x_\alpha^k + \frac{g^{ik}}{4} h_{\alpha\gamma} U_{(k)j}^{(\gamma)}, & p \geq 2. \end{cases}$$

The main result of the metrical multi-time Lagrange geometry is the theorem of existence of the *Cartan canonical  $h$ -normal linear connection*  $CT$  which allow the subsequent development of the *metrical multi-time Lagrangian theory of physical fields*.

**Theorem 2.4** (of existence and uniqueness of Cartan canonical connection) *On the metrical multi-time Lagrange space  $ML_p^n = (J^1(T, M), L)$  endowed with its canonical nonlinear connection  $\Gamma$ , there is a unique  $h$ -normal  $\Gamma$ -linear connection*

$$C\Gamma = (H_{\alpha\beta}^\gamma, G_{j\gamma}^k, L_{jk}^i, C_{j(k)}^{i(\gamma)})$$

having the metrical properties

$$i) \quad g_{ij|k} = 0, \quad g_{ij}|_{(k)}^{(\gamma)} = 0,$$

$$ii) \quad G_{j\gamma}^k = \frac{g^{ki}}{2} \frac{\delta g_{ij}}{\delta t^\gamma}, \quad L_{ij}^k = L_{ji}^k, \quad C_{j(k)}^{i(\gamma)} = C_{k(j)}^{i(\gamma)}.$$

Moreover, the coefficients  $L_{jk}^i$  and  $C_{j(k)}^{i(\gamma)}$  of the Cartan canonical connection have the expressions [12]

$$(2.10) \quad \begin{aligned} L_{jk}^i &= \frac{g^{im}}{2} \left( \frac{\delta g_{mj}}{\delta x^k} + \frac{\delta g_{mk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^m} \right), \\ C_{j(k)}^{i(\gamma)} &= \frac{g^{im}}{2} \left( \frac{\partial g_{mj}}{\partial x_\gamma^k} + \frac{\partial g_{mk}}{\partial x_\gamma^j} - \frac{\partial g_{jk}}{\partial x_\gamma^m} \right). \end{aligned}$$

**Remarks 2.5** i) In the particular case  $(T, h) = (R, \delta)$ , the Cartan canonical  $\delta$ -normal  $\Gamma$ -linear connection of the relativistic rheonomic Lagrange space  $RL^n = (J^1(R, M), L)$  reduces to the Cartan canonical connection used in [7].

ii) As a rule, the Cartan canonical connection of a metrical multi-time Lagrange space  $ML_p^n$  verifies also the metrical properties

$$h_{\alpha\beta/\gamma} = h_{\alpha\beta|k} = h_{\alpha\beta}|_{(k)}^{(\gamma)} = 0 \text{ and } g_{ij/\gamma} = 0.$$

iii) In the case  $p = \dim T \geq 2$ , the coefficients of the Cartan connection of a metrical multi-time Lagrange space reduce to

$$\bar{G}_{\alpha\beta}^\gamma = H_{\alpha\beta}^\gamma, \quad G_{j\gamma}^k = \frac{g^{ki}}{2} \frac{\partial g_{ij}}{\partial t^\gamma}, \quad L_{jk}^i = \Gamma_{jk}^i, \quad C_{j(k)}^{i(\gamma)} = 0.$$

**Theorem 2.5** *The torsion  $d$ -tensor  $\mathbf{T}$  of the Cartan canonical connection of a metrical multi-time Lagrange space is determined by the local components*

$$(2.11) \quad \begin{array}{c|cc|cc|cc} & \hline & h_T & & h_M & & v \\ & p=1 & p \geq 2 & p=1 & p \geq 2 & p=1 & p \geq 2 \\ \hline h_T h_T & 0 & 0 & 0 & 0 & 0 & R_{(\mu)\alpha\beta}^{(m)} \\ h_M h_T & 0 & 0 & T_{1j}^m & T_{\alpha j}^m & R_{(1)1j}^{(m)} & R_{(\mu)\alpha j}^{(m)} \\ h_M h_M & 0 & 0 & 0 & 0 & R_{(1)ij}^{(m)} & R_{(\mu)ij}^{(m)} \\ v h_T & 0 & 0 & 0 & 0 & P_{(1)1(j)}^{(m)(1)} & P_{(\mu)\alpha(j)}^{(m)(\beta)} \\ v h_M & 0 & 0 & P_{i(j)}^{m(1)} & 0 & P_{(1)i(j)}^{(m)(1)} & 0 \\ vv & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

where,



i) for  $p = \dim T = 1$ , we have

$$\begin{aligned} T_{1j}^m &= -G_{j1}^m, \quad P_{i(j)}^{m(1)} = C_{i(j)}^{m(1)}, \quad P_{(1)1(j)}^{(m)(1)} = -G_{j1}^m, \\ P_{(1)i(j)}^{(m)(1)} &= \frac{\partial N_{(1)i}^{(m)}}{\partial y^j} - L_{ji}^m, \quad \frac{\delta N_{(1)i}^{(m)}}{\delta x^j} - \frac{\delta N_{(1)j}^{(m)}}{\delta x^i}, \\ R_{(1)1j}^{(m)} &= -\frac{\partial N_{(1)j}^{(m)}}{\partial t} + H_{11}^1 \left[ N_{(1)j}^{(m)} - \frac{\partial N_{(1)j}^{(m)}}{\partial y^k} y^k \right]; \end{aligned}$$

ii) for  $p = \dim T \geq 2$ , denoting

$$\begin{aligned} F_{i(\mu)}^m &= \frac{g^{mp}}{2} \left[ \frac{\partial g_{pi}}{\partial t^\mu} + \frac{1}{2} h_{\mu\beta} U_{(p)i}^{(\beta)} \right], \\ H_{\mu\alpha\beta}^\gamma &= \frac{\partial H_{\mu\alpha}^\gamma}{\partial t^\beta} - \frac{\partial H_{\mu\beta}^\gamma}{\partial t^\alpha} + H_{\mu\alpha}^\eta H_{\eta\beta}^\gamma - H_{\mu\beta}^\eta H_{\eta\alpha}^\gamma, \\ r_{pij}^m &= \frac{\partial \Gamma_{pi}^m}{\partial x^j} - \frac{\partial \Gamma_{pj}^m}{\partial x^i} + \Gamma_{pi}^k \Gamma_{kj}^m - \Gamma_{pj}^k \Gamma_{ki}^m, \end{aligned}$$

we have

$$\begin{aligned} T_{\alpha j}^m &= -G_{j\alpha}^m, \quad P_{(\mu)\alpha(j)}^{m(\beta)} = -\delta_\gamma^\beta G_{j\alpha}^m, \quad R_{(\mu)\alpha(j)}^{(m)} = -H_{\mu\alpha\beta}^\gamma x_\gamma^m, \\ R_{(\mu)\alpha j}^{(m)} &= -\frac{\partial N_{(\mu)j}^{(m)}}{\partial t^\alpha} + \frac{g^{mk}}{2} H_{\mu\alpha}^\beta \left[ \frac{\partial g_{jk}}{\partial t^\beta} + \frac{h_{\beta\gamma}}{2} U_{(k)j}^{(\gamma)} \right], \\ R_{(\mu)ij}^{(m)} &= r_{ijk}^m x_\mu^k + \left[ F_{i(\mu)|j}^m - F_{j(\mu)|i}^m \right]. \end{aligned}$$

**Remark 2.6** In the case of autonomous metrical multi-time Lagrange space of electrodynamics (i. e.,  $g_{ij}(t^\gamma, x^k, x_\gamma^k) = g_{ij}(x^k)$ ), all torsion d-tensors of the Cartan connection vanish, except

$$\begin{aligned} R_{(\mu)\alpha\beta}^{(m)} &= -H_{\mu\alpha\beta}^\gamma x_\gamma^m, \quad R_{(\mu)\alpha j}^{(m)} = -\frac{h_{\mu\eta} g^{mk}}{4} \left[ H_{\alpha\gamma}^\eta U_{(k)j}^{(\gamma)} + \frac{\partial U_{(k)j}^{(\eta)}}{\partial t^\alpha} \right], \\ R_{(\mu)ij}^{(m)} &= r_{ijk}^m x_\mu^k + \frac{h_{\mu\eta} g^{mk}}{4} \left[ U_{(k)i|j}^{(\eta)} + U_{(k)j|i}^{(\eta)} \right], \end{aligned}$$

where  $H_{\mu\alpha\beta}^\gamma$  (resp.  $r_{ijk}^m$ ) are the curvature tensors of the semi-Riemannian metric  $h_{\alpha\beta}$  (resp.  $g_{ij}$ ).

**Theorem 2.6** The curvature d-tensor  $\mathbf{R}$  of the Cartan canonical connection is de-

terminated by the local components

	$h_T$		$h_M$		$v$	
	$p = 1$	$p \geq 2$	$p = 1$	$p \geq 2$	$p = 1$	$p \geq 2$
$h_T h_T$	$0$	$H_{\eta\beta\gamma}^\alpha$	$0$	$R_{i\beta\gamma}^l$	$0$	$R_{(\eta)(i)\beta\gamma}^{(l)(\alpha)}$
$h_M h_T$	$0$	$0$	$R_{i1k}^l$	$R_{i\beta k}^l$	$R_{(1)(i)1k}^{(l)(1)} = R_{i1k}^l$	$R_{(\eta)(i)\beta k}^{(l)(\alpha)}$
$h_M h_M$	$0$	$0$	$R_{ijk}^l$	$R_{ijk}^l$	$R_{(1)(i)jk}^{(l)(1)} = R_{ijk}^l$	$R_{(\eta)(i)jk}^{(l)(\alpha)}$
$vh_T$	$0$	$0$	$P_{i1(k)}^{(l)(1)}$	$0$	$P_{(1)(i)1(k)}^{(l)(1)(1)} = P_{i1(k)}^{(l)(1)}$	$0$
$vh_M$	$0$	$0$	$P_{ij(k)}^{l(1)}$	$0$	$P_{(1)(i)j(k)}^{(l)(1)(1)} = P_{ij(k)}^{l(1)}$	$0$
$vv$	$0$	$0$	$S_{i(j)(k)}^{l(1)(1)}$	$0$	$S_{(1)(i)(j)(k)}^{(l)(1)(1)(1)} = S_{i(j)(k)}^{l(1)(1)}$	$0$

where  $R_{(\eta)(i)\beta\gamma}^{(l)(\alpha)} = \delta_\eta^\alpha R_{i\beta\gamma}^l + \delta_i^l H_{\eta\beta\gamma}^\alpha$ ,  $R_{(\eta)(i)\beta k}^{(l)(\alpha)} = \delta_\eta^\alpha R_{i\beta k}^l$ ,  $R_{(\eta)(i)jk}^{(l)(\alpha)} = \delta_\eta^\alpha R_{ijk}^l$  and

i) for  $p = \dim T = 1$ , we have

$$R_{i1k}^l = \frac{\delta G_{i1}^l}{\delta x^k} - \frac{\delta L_{ik}^l}{\delta t} + G_{i1}^m L_{mk}^l - L_{ik}^m G_{m1}^l + C_{i(m)}^{l(1)} R_{(1)1k}^{(m)},$$

$$R_{ijk}^l = \frac{\delta L_{ij}^l}{\delta x^k} - \frac{\delta L_{ik}^l}{\delta x^j} + L_{ij}^m L_{mk}^l - L_{ik}^m L_{mj}^l + C_{i(m)}^{l(1)} R_{(1)jk}^{(m)},$$

$$P_{i1(k)}^{l(1)} = \frac{\partial G_{i1}^l}{\partial y^k} - C_{i(k)/1}^{l(1)} + C_{i(m)}^{l(1)} P_{(1)1(k)}^{(m)(1)},$$

$$P_{ij(k)}^{l(1)} = \frac{\partial L_{ij}^l}{\partial y^k} - C_{i(k)|j}^{l(1)} + C_{i(m)}^{l(1)} P_{(1)j(k)}^{(m)(1)},$$

$$S_{i(j)(k)}^{l(1)(1)} = \frac{\partial C_{i(j)}^{l(1)}}{\partial y^k} - \frac{\partial C_{i(k)}^{l(1)}}{\partial y^j} + C_{i(j)}^{m(1)} C_{m(k)}^{l(1)} - C_{i(k)}^{m(1)} C_{m(j)}^{l(1)};$$

ii) for  $p = \dim T \geq 2$ , we have

$$H_{\eta\beta\gamma}^\alpha = \frac{\partial H_{\eta\beta}^\alpha}{\partial t^\gamma} - \frac{\partial H_{\eta\gamma}^\alpha}{\partial t^\beta} + H_{\eta\beta}^\mu H_{\mu\gamma}^\alpha - H_{\eta\gamma}^\mu H_{\mu\beta}^\alpha,$$

$$R_{i\beta\gamma}^l = \frac{\delta G_{i\beta}^l}{\delta t^\gamma} - \frac{\delta G_{i\gamma}^l}{\delta t^\beta} + G_{i\beta}^m G_{m\gamma}^l - G_{i\gamma}^m G_{m\beta}^l,$$

$$R_{i\beta k}^l = \frac{\delta G_{i\beta}^l}{\delta x^k} - \frac{\delta \Gamma_{ik}^l}{\delta t^\beta} + G_{i\beta}^m \Gamma_{mk}^l - \Gamma_{ik}^m G_{m\beta}^l,$$

$$R_{ijk}^l = r_{ijk}^l = \frac{\partial \Gamma_{ij}^l}{\partial x^k} - \frac{\partial \Gamma_{ik}^l}{\partial x^j} + \Gamma_{ij}^m \Gamma_{mk}^l - \Gamma_{ik}^m \Gamma_{mj}^l.$$

**Remark 2.7** In the case of an autonomous metrical multi-time Lagrange space of electrodynamics (i. e. ,  $g_{ij}(t^\gamma, x^k, x_\gamma^k) = g_{ij}(x^k)$ ), all curvature d-tensors of the Cartan canonical connection vanish, except  $H_{\eta\beta\gamma}^\alpha$  and  $R_{ijk}^l = r_{ijk}^l$ , that is, the curvature tensors of the semi-Riemannian metrics  $h_{\alpha\beta}$  and  $g_{ij}$ .

### 3 Electromagnetic field. Maxwell equations

Let  $ML_p^n = (J^1(T, M), L)$  be a metrical multi-time Lagrange space and  $\Gamma = (M_{(\alpha)\beta}^{(i)}, N_{(\alpha)j}^{(i)})$  its canonical nonlinear connection. Let us consider  $CT = (H_{\alpha\beta}^\gamma, G_{i\gamma}^k, L_{ij}^k, C_{i(j)}^{k(\gamma)})$  the Cartan canonical connection of  $ML_p^n$ .

Using the *canonical Liouville d-tensor*  $\mathbf{C} = x_\alpha^i \frac{\partial}{\partial x_\alpha^i}$  and the fundamental vertical metrical d-tensor  $G_{(i)(k)}^{(\alpha)(\beta)}$  of the metrical multi-time Lagrange space  $ML_p^n$ , we construct the *metrical deflection d-tensors*

$$\begin{aligned} \bar{D}_{(i)\beta}^{(\alpha)} &= G_{(i)(k)}^{(\alpha)(\gamma)} \bar{D}_{(\gamma)\beta}^{(k)} = x_{(i)/\beta}^{(\alpha)}, \\ D_{(i)j}^{(\alpha)} &= G_{(i)(k)}^{(\alpha)(\gamma)} D_{(\gamma)j}^{(k)} = x_{(i)|j}^{(\alpha)}, \\ d_{(i)(j)}^{(\alpha)(\beta)} &= G_{(i)(k)}^{(\alpha)(\gamma)} d_{(\gamma)(j)}^{(k)(\beta)} = x_{(i)|j}^{(\alpha)(\beta)}, \end{aligned} \quad (3.12)$$

where  $x_{(i)}^{(\alpha)} = G_{(i)(k)}^{(\alpha)(\gamma)} x_\gamma^k$  and  $_{/\beta}$ ,  $_{|j}$  and  $_{(j)}^{(\beta)}$  are the local covariant derivatives induced by  $CT$ .

Taking into account the expressions of the local covariant derivatives of  $CT$  (see the papers [10], [13]), by a direct calculation, we obtain

**Proposition 3.1** *The metrical deflection d-tensors of a metrical multi-time Lagrange space  $ML_p^n$  have the expressions:*

i) for  $p = 1$ ,

$$\begin{aligned} \bar{D}_{(i)1}^{(1)} &= \frac{h^{11}}{2} \frac{\delta g_{im}}{\delta t} y^m, \\ D_{(i)j}^{(1)} &= h^{11} g_{ik} \left[ -N_{(1)j}^{(k)} + L_{jm}^k y^m \right], \\ d_{(i)(j)}^{(1)(1)} &= h^{11} \left[ g_{ij} + g_{ik} C_{m(j)}^{k(1)} y^m \right]; \end{aligned} \quad (3.13)$$

ii) for  $p \geq 2$ ,

$$\begin{aligned} \bar{D}_{(i)\beta}^{(\alpha)} &= \frac{h^{\alpha\gamma}}{2} \frac{\partial g_{km}}{\partial t^\beta} x_\gamma^m, \\ D_{(i)j}^{(\alpha)} &= -\frac{h^{\alpha\gamma}}{2} \frac{\partial g_{ij}}{\partial t^\gamma} - \frac{1}{4} U_{(i)j}^{(\alpha)}, \\ d_{(i)(j)}^{(\alpha)(\beta)} &= h^{\alpha\beta} g_{ij}. \end{aligned} \quad (3.14)$$

**Remark 3.1** In the particular case of an autonomous metrical multi-time Lagrange space of electrodynamics (i. e.,  $g_{ij} = g_{ij}(x^k)$ ), we have

$$\bar{D}_{(i)\beta}^{(\alpha)} = 0, \quad D_{(i)j}^{(\alpha)} = -\frac{1}{4} U_{(i)j}^{(\alpha)}, \quad d_{(i)(j)}^{(\alpha)(\beta)} = h^{\alpha\beta} g_{ij}.$$

In order to construct the metrical multi-time Lagrangian theory of electromagnetism, we introduce the following

**Definition 3.1** The distinguished 2-form on  $J^1(T, M)$ ,

$$F = F_{(i)j}^{(\alpha)} \delta x_\alpha^i \wedge dx^j + f_{(i)(j)}^{(\alpha)(\beta)} \delta x_\alpha^i \wedge \delta x_\beta^j, \quad (3.15)$$

where  $F_{(i)j}^{(\alpha)} = \frac{1}{2} [D_{(i)j}^{(\alpha)} - D_{(j)i}^{(\alpha)}]$  and  $f_{(i)(j)}^{(\alpha)(\beta)} = \frac{1}{2} [d_{(i)(j)}^{(\alpha)(\beta)} - d_{(j)(i)}^{(\alpha)(\beta)}]$ , is called the *electromagnetic d-form* of the metrical multi-time Lagrange space  $ML_p^n$ .

**Remark 3.2** The naturalness of the previous definition comes considering the particular case of a relativistic rheonomic Lagrange space (i. e.,  $(T, h) = (R, \delta)$ ). In this case, we recover the electromagnetic d-tensor of the Miron-Anastasiu electro-magnetism [7].

By simple computations, we find

**Proposition 3.2** *The components  $F_{(i)j}^{(\alpha)}$  and  $f_{(i)(j)}^{(\alpha)(\beta)}$  of the electromagnetic d-form  $F$  of a metrical multi-time Lagrange space are described by the formulas:*

i) in the case  $p = 1$ ,

$$F_{(i)j}^{(1)} = \frac{h^{11}}{2} [g_{jm} N_{(1)i}^{(m)} - g_{im} N_{(1)j}^{(m)} + (g_{ik} L_{jm}^k - g_{jk} L_{im}^k) y^m], \quad f_{(i)(j)}^{(1)(1)} = 0;$$

ii) in the case  $p \geq 2$ ,

$$F_{(i)j}^{(\alpha)} = \frac{1}{8} [U_{(j)i}^{(\alpha)} - U_{(i)j}^{(\alpha)}], \quad f_{(i)(j)}^{(\alpha)(\beta)} = 0.$$

**Remark 3.3** We emphasize that, in the particular case of an autonomous metrical multi-time Lagrange space (i. e.  $g_{ij} = g_{ij}(x^k)$ ), the electromagnetic components get the expressions

$$F_{(i)j}^{(\alpha)} = \frac{1}{8} [U_{(j)i}^{(\alpha)} - U_{(i)j}^{(\alpha)}], \quad f_{(i)(j)}^{(\alpha)(\beta)} = 0$$

The main result of the electromagnetic metrical multi-time Lagrangian theory is the following

**Theorem 3.3** *The electromagnetic components  $F_{(i)j}^{(\alpha)}$  of a metrical multi-time Lagrange space  $ML_p^n = (J^1(T, M), L)$  are governed by the Maxwell equations:*

i) for  $p = 1$ ,

$$\left\{ \begin{array}{l} F_{(i)k/1}^{(1)} = \frac{1}{2} \mathcal{A}_{\{i,k\}} \left\{ \bar{D}_{(i)1|k}^{(1)} + D_{(i)m}^{(1)} T_{1k}^m + d_{(i)(m)}^{(1)(1)} R_{(1)1k}^{(m)} - [T_{1i|k}^p + C_{k(m)}^{p(1)} R_{(1)1i}^{(m)}] y_{(p)} \right\} \\ \sum_{\{i,j,k\}} F_{(i)j|k}^{(1)} = -\frac{1}{2} \sum_{\{i,j,k\}} C_{(i)(l)(m)}^{(1)(1)(1)} R_{(1)jk}^{(m)} y^l \\ \sum_{\{i,j,k\}} F_{(i)j|k}^{(1)}|_{(k)}^{(1)} = 0, \end{array} \right.$$

ii) for  $p \geq 2$ ,

$$\left\{ \begin{array}{l} F_{(i)k/\beta}^{(\alpha)} = \frac{1}{2} \mathcal{A}_{\{i,k\}} \left\{ \bar{D}_{(i)\beta|k}^{(\alpha)} + D_{(i)m}^{(\alpha)} T_{\beta k}^m + d_{(i)(m)}^{(\alpha)(\mu)} R_{(\mu)\beta k}^{(m)} - [T_{\beta i|k}^p + C_{k(m)}^{p(\mu)} R_{(\mu)\beta i}^{(m)}] x_{(p)}^{(\alpha)} \right\} \\ \sum_{\{i,j,k\}} F_{(i)j|k}^{(\alpha)} = 0 \\ \sum_{\{i,j,k\}} F_{(i)j|k}^{(\alpha)}|_{(k)}^{(\gamma)} = 0, \end{array} \right.$$

where  $y_{(p)} = G_{(p)(q)}^{(1)(1)} y^q$ ,  $C_{(i)(l)(m)}^{(1)(1)(1)} = G_{(l)(q)}^{(1)(1)} C_{i(m)}^{q(1)} = \frac{h^{11}}{2} \frac{\partial^3 L}{\partial y^i \partial y^l \partial y^m}$ ,  $x_{(p)}^{(\alpha)} = G_{(p)(q)}^{(\alpha)(\beta)} x_{\beta}^q$ .

**Proof.** Firstly, we point out that the Ricci identities [13] applied to the spatial metrical d-tensor  $g_{ij}$  imply that the following curvature d-tensor identities

$$R_{mi\beta k} + R_{im\beta k} = 0, \quad R_{mijk} + R_{imjk} = 0, \quad P_{mij(k)}^{(\gamma)} + P_{imj(k)}^{(\gamma)} = 0,$$

where  $R_{mi\beta k} = g_{ip} R_{m\beta k}^p$ ,  $R_{mijk} = g_{ip} R_{mj k}^p$  and  $P_{mij(k)}^{(\gamma)} = g_{ip} P_{mj(k)}^{p(\gamma)}$ , are true.

Now, let us consider the following general deflection d-tensor identities [13]

$$d_1) \quad \bar{D}_{(\nu)\beta|k}^{(p)} - D_{(\nu)k/\beta}^{(p)} = x_{\nu}^m R_{m\beta k}^p - D_{(\nu)m}^{(p)} T_{\beta k}^m - d_{(\nu)(m)}^{(p)(\mu)} R_{(\mu)\beta k}^{(m)},$$

$$d_2) \quad D_{(\nu)j|k}^{(p)} - D_{(\nu)k|j}^{(p)} = x_{\nu}^m R_{mj k}^p - d_{(\nu)(m)}^{(p)(\mu)} R_{(\mu)j k}^{(m)},$$

$$d_3) \quad D_{(\nu)j|k}^{(p)(\gamma)} - d_{(\nu)(k)|j}^{(p)(\gamma)} = x_{\nu}^m P_{mj(k)}^{p(\gamma)} - D_{(\nu)m}^{(p)} C_{j(k)}^{m(\gamma)} - d_{(\nu)(m)}^{(p)(\mu)} P_{(\mu)j(k)}^{(m)(\gamma)},$$

where  $\bar{D}_{(\alpha)\beta}^{(i)} = x_{\alpha/\beta}^i$ ,  $D_{(\alpha)j}^{(i)} = x_{\alpha|j}^i$ ,  $d_{(\alpha)(j)}^{(i)(\beta)} = x_{\alpha}^i |_{(j)}^{(\beta)}$ . Contracting the deflection d-tensor identities by  $G_{(i)(p)}^{(\alpha)(\nu)}$  and using the above curvature d-tensor equalities, we obtain the metrical deflection d-tensors identities:

$$d'_1) \quad \bar{D}_{(i)\beta|k}^{(\alpha)} - D_{(i)k/\beta}^{(\alpha)} = -x_{(m)}^{(\alpha)} R_{i\beta k}^m - D_{(i)m}^{(\alpha)} T_{\beta k}^m - d_{(i)(m)}^{(\alpha)(\mu)} R_{(\mu)\beta k}^{(m)},$$

$$d'_2) \quad D_{(i)j|k}^{(\alpha)} - D_{(i)k|j}^{(\alpha)} = -x_{(m)}^{(\alpha)} R_{ij k}^m - d_{(i)(m)}^{(\alpha)(\mu)} R_{(\mu)j k}^{(m)},$$

$$d'_3) \quad D_{(i)j|k}^{(\alpha)(\gamma)} - d_{(i)(k)|j}^{(\alpha)(\gamma)} = -x_{(m)}^{(\alpha)} P_{ij(k)}^{m(\gamma)} - D_{(i)m}^{(\alpha)} C_{j(k)}^{m(\gamma)} - d_{(i)(m)}^{(\alpha)(\mu)} P_{(\mu)j(k)}^{(m)(\gamma)}.$$

At the same time, we recall that the following Bianchi identities [10]

$$b_1) \quad \mathcal{A}_{\{j,k\}} \left\{ R_{j\alpha k}^l + T_{\alpha j|k}^l + C_{k(m)}^{l(\mu)} R_{(\mu)\alpha j}^{(m)} \right\} = 0,$$

$$b_2) \quad \sum_{\{i,j,k\}} \left\{ R_{ijk}^l - C_{k(m)}^{l(\mu)} R_{(\mu)ij}^{(m)} \right\} = 0,$$

$$b_3) \quad \mathcal{A}_{\{j,k\}} \left\{ P_{jk(p)}^{l(\varepsilon)} + C_{j(p)|k}^{l(\varepsilon)} + C_{k(m)}^{l(\mu)} P_{(\mu)j(p)}^{(m)(\varepsilon)} \right\} = 0,$$

where  $\mathcal{A}_{\{j,k\}}$  means alternate sum and  $\sum_{\{i,j,k\}}$  means cyclic sum, hold good.

In order to obtain the first Maxwell identity, we permute  $i$  and  $k$  in  $d'_1$  and we subtract the new identity from the initial one. Finally, using the Bianchi identity  $b_1$ , we obtain what we were looking for.

Doing a cyclic sum by the indices  $\{i, j, k\}$  in  $d'_2$  and using the Bianchi identity  $b_2$ , it follows the second Maxwell equation.

Applying a Christoffel process to the indices  $\{i, j, k\}$  in  $d'_3$  and combining with the Bianchi identity  $b_3$  and the relation  $P_{(\mu)j(p)}^{(m)(\varepsilon)} = P_{(\mu)p(j)}^{(m)(\varepsilon)}$ , we get a new identity. The cyclic sum by the indices  $\{i, j, k\}$  applied to this last identity implies the third Maxwell equation. ■

**Remark 3.4** In the case of an autonomous metrical multi-time Lagrange space of electrodynamics (i. e.,  $g_{ij} = g_{ij}(x^k)$ ), the Maxwell equations take a more simple form, namely,

$$F_{(i)k/\beta}^{(\alpha)} = \frac{1}{2} \mathcal{A}_{\{i,k\}} h^{\alpha\mu} g_{im} R_{(\mu)\beta k}^{(m)}, \quad \sum_{\{i,j,k\}} F_{(i)j|k}^{(\alpha)} = 0, \quad \sum_{\{i,j,k\}} F_{(ij)|k}^{(\alpha)(\gamma)} = 0.$$

## 4 Gravitational field. Einstein equations

Let  $h = (h_{\alpha\beta})$  be a fixed semi-Riemannian metric on the temporal manifold  $T$  and  $\Gamma = (M_{(\alpha)\beta}^{(i)}, N_{(\alpha)j}^{(i)})$  a fixed nonlinear connection on the 1-jet space  $J^1(T, M)$ . In order to develop the metrical multi-time Lagrange theory of gravitational field, we introduce the following

**Definition 4.1** From physical point of view, an adapted metrical d-tensor  $G$  on  $E = J^1(T, M)$ , expressed locally by

$$G = h_{\alpha\beta} dt^\alpha \otimes dt^\beta + g_{ij} dx^i \otimes dx^j + h^{\alpha\beta} g_{ij} \delta x_\alpha^i \otimes \delta x_\beta^j,$$

where  $g_{ij} = g_{ij}(t^\gamma, x^k, x_\gamma^k)$  is a d-tensor field on  $E$ , symmetric, of rank  $n = \dim M$  and having a constant signature on  $E$ , is called a *gravitational h-potential*.

**Remark 4.1** The naturalness of this definition comes from the particular case  $(T, h) = (R, \delta)$ . In this case, we recover the *gravitational potentials*  $g_{ij}(x, y)$  from Miron-Anastasiu theory of gravitational field [7].

Now, taking  $ML_p^n = (J^1(T, M), L)$  a metrical multi-time Lagrange space, via its fundamental vertical metrical d-tensor

$$(4.1) \quad G_{(i)(j)}^{(\alpha)(\beta)} = \frac{1}{2} \frac{\partial^2 L}{\partial x_\alpha^i \partial x_\beta^j} = \begin{cases} h^{11}(t) g_{ij}(t, x^k, y^k), & p = \dim T = 1 \\ h^{\alpha\beta}(t^\gamma) g_{ij}(t^\gamma, x^k), & p = \dim T \geq 2, \end{cases}$$

and its canonical nonlinear connection  $\Gamma = (M_{(\alpha)\beta}^{(i)}, N_{(\alpha)j}^{(i)})$ , one induces a natural gravitational  $h$ -potential, setting

$$G = h_{\alpha\beta} dt^\alpha \otimes dt^\beta + g_{ij} dx^i \otimes dx^j + h^{\alpha\beta} g_{ij} \delta x_\alpha^i \otimes \delta x_\beta^j.$$

Let us consider  $CT = (H_{\alpha\beta}^\gamma, G_{j\gamma}^k, L_{jk}^i, C_{j(k)}^{i(\gamma)})$  the Cartan canonical connection of  $ML_p^n$ .

We postulate that the Einstein which govern the gravitational  $h$ -potential  $G$  of the metrical multi-time Lagrange space  $ML_p^n$  are the Einstein equations attached to the Cartan canonical connection  $CT$  of  $ML_p^n$  and the adapted metric  $G$  on  $E$ , that is,

$$(4.2) \quad Ric(CT) - \frac{Sc(CT)}{2} G = \mathcal{K}T,$$

where  $Ric(CT)$  represents the Ricci d-tensor of the Cartan connection,  $Sc(CT)$  is its scalar curvature,  $\mathcal{K}$  is the Einstein constant and  $T$  is an intrinsic tensor of matter which is called the *stress-energy* d-tensor.

In the adapted basis  $(X_A) = \left( \frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x_\alpha^i} \right)$  of the nonlinear connection  $\Gamma$  of  $ML_p^n$ , the curvature d-tensor  $\mathbf{R}$  of the Cartan connection is expressed locally by  $\mathbf{R}(X_C, X_B)X_A = R_{ABC}^D X_D$ . It follows that we have  $R_{AB} = Ric(X_A, X_B) = R_{ABD}^D$  and  $Sc(CT) = G^{AB} R_{AB}$ , where

$$(4.3) \quad G^{AB} = \begin{cases} h_{\alpha\beta}, & \text{for } A = \alpha, B = \beta \\ g^{ij}, & \text{for } A = i, B = j \\ h_{\alpha\beta} g^{ij}, & \text{for } A = \binom{i}{\alpha}, B = \binom{j}{\beta} \\ 0, & \text{otherwise.} \end{cases}$$

Taking into account, on the one hand, the form of the fundamental vertical metrical d-tensor  $G_{(i)(j)}^{(\alpha)(\beta)}$  of the metrical multi-time Lagrange space  $ML_p^n$ , and, on the other hand, the expressions of local curvature d-tensors attached to the Cartan canonical connection  $CT$ , by a direct calculation, we deduce

**Theorem 4.1** *The Ricci d-tensor  $Ric(CT)$  of the Cartan canonical connection  $CT$  of a metrical multi-time Lagrange space, is determined by the following components:*

i) for  $p = \dim T = 1$ ,

$$\begin{aligned} R_{11} \stackrel{not}{=} H_{11} = 0, \quad R_{i1} = R_{i1m}^m, \quad R_{ij} = R_{ijm}^m, \quad R_{i(j)}^{(1)} \stackrel{not}{=} P_{i(j)}^{(1)} = -P_{im(j)}^m, \\ R_{(i)j}^{(1)} \stackrel{not}{=} P_{(i)j}^{(1)} = P_{ij(m)}^m, \quad R_{(i)1}^{(1)} \stackrel{not}{=} P_{(i)1}^{(1)} = P_{i1(m)}^m, \quad R_{(i)(j)}^{(1)(1)} \stackrel{not}{=} S_{(i)(j)}^{(1)(1)} = S_{i(j)(m)}^{m(1)(1)}; \end{aligned}$$

ii) for  $p = \dim T \geq 2$ ,

$$\begin{aligned} R_{(\alpha)(\beta)} \stackrel{not}{=} H_{\alpha\beta} = H_{\alpha\beta\mu}^\mu, \quad R_{i\alpha} = R_{i\alpha m}^m, \quad R_{ij} = R_{ijm}^m, \quad R_{i(j)}^{(\alpha)} \stackrel{not}{=} P_{i(j)}^{(\alpha)} = 0, \\ R_{(i)j}^{(\alpha)} \stackrel{not}{=} P_{(i)j}^{(\alpha)} = 0, \quad R_{(i)\beta}^{(\alpha)} \stackrel{not}{=} P_{(i)\beta}^{(\alpha)} = 0, \quad R_{(i)(j)}^{(\alpha)(\beta)} \stackrel{not}{=} S_{(i)(j)}^{(\alpha)(\beta)} = 0. \end{aligned}$$

Denoting  $H = h^{\alpha\beta} H_{\alpha\beta}$ ,  $R = g^{ij} R_{ij}$  and  $S = h_{\alpha\beta} g^{ij} S_{(i)(j)}^{(\alpha)(\beta)}$ , it follows

**Corollary 4.2** *The scalar curvature of  $Sc(CT)$  of the Cartan canonical connection  $CT$  of a metrical multi-time Lagrange space, is given by the formulas*

i) for  $p = \dim T = 1$ ,  $Sc(CT) = R + S$ ;

ii) for  $p = \dim T \geq 2$ ,  $Sc(CT) = H + R$ .

**Remark 4.2** In the particular case of an autonomous metrical multi-time Lagrange space of electrodynamics (i. e.,  $g_{ij} = g_{ij}(x^k)$ ), all Ricci d-tensor components vanish, except  $H_{\alpha\beta}$  and  $R_{ij} = r_{ij}$ , where  $H_{\alpha\beta}$  (resp.  $r_{ij}$ ) are the local Ricci tensors associated to the semi-Riemannian metric  $h_{\alpha\beta}$  (resp.  $g_{ij}$ ). It follows that the scalar curvature of a this space is  $Sc(CT) = H + r$ , where  $H$  and  $r$  are the scalar curvatures of the semi-Riemannian metrics  $h_{\alpha\beta}$  and  $g_{ij}$ .

The main result of the metrical multi-time Lagrange theory of gravitational field is given by the following

**Theorem 4.3** *The Einstein equations which govern the gravitational h-potential  $G$  induced by the Kronecker h-regular Lagrangian of a metrical multi-time Lagrange space  $ML_p^n$ , take the form*

$$(E_1) \quad \begin{cases} -\frac{R+S}{2} h_{11} = \mathcal{K}T_{11} \\ R_{ij} - \frac{R+S}{2} g_{ij} = \mathcal{K}T_{ij} \\ S_{(i)(j)}^{(1)(1)} - \frac{R+S}{2} h^{11} g_{ij} = \mathcal{K}T_{(i)(j)}^{(1)(1)}, \end{cases}$$

$$(E_2) \quad \begin{cases} 0 = \mathcal{T}_{1i}, & R_{i1} = \mathcal{K}\mathcal{T}_{i1}, & P_{(i)1}^{(1)} = \mathcal{K}\mathcal{T}_{(i)1}^{(1)} \\ 0 = \mathcal{T}_{1(i)}, & P_{i(j)}^{(1)} = \mathcal{K}\mathcal{T}_{i(j)}^{(1)}, & P_{(i)j}^{(1)} = \mathcal{K}\mathcal{T}_{(i)j}^{(1)}, \end{cases}$$

i) for  $p = \dim T \geq 2$ ,

$$(E_1) \quad \begin{cases} H_{\alpha\beta} - \frac{H+R}{2}h_{\alpha\beta} = \mathcal{K}\mathcal{T}_{\alpha\beta} \\ R_{ij} - \frac{H+R}{2}g_{ij} = \mathcal{K}\mathcal{T}_{ij} \\ -\frac{H+R}{2}h^{\alpha\beta}g_{ij} = \mathcal{K}\mathcal{T}_{(i)(j)}^{(\alpha)(\beta)}, \end{cases}$$

$$(E_2) \quad \begin{cases} 0 = \mathcal{T}_{\alpha i}, & R_{i\alpha} = \mathcal{K}\mathcal{T}_{i\alpha}, & 0 = \mathcal{T}_{(i)\beta}^{(\alpha)} \\ 0 = \mathcal{T}_{\alpha(i)}^{(\beta)}, & 0 = \mathcal{T}_{i(j)}^{(\alpha)}, & 0 = \mathcal{T}_{(i)j}^{(\alpha)}, \end{cases}$$

where  $\mathcal{T}_{AB}$ ,  $A, B \in \{\alpha, i, \binom{(\alpha)}{(i)}\}$  are the adapted local components of the stress-energy d-tensor  $\mathcal{T}$ .

**Remarks 4.3** i) Assuming that  $p = \dim T > 2$  and  $n = \dim M > 2$ , the set  $(E_1)$  of the Einstein equations can be rewritten in the more natural form

$$(E'_1) \quad \begin{cases} H_{\alpha\beta} - \frac{H}{2}h_{\alpha\beta} = \mathcal{K}\tilde{\mathcal{T}}_{\alpha\beta} \\ R_{ij} - \frac{R}{2}g_{ij} = \mathcal{K}\tilde{\mathcal{T}}_{ij}, \end{cases}$$

where  $\tilde{\mathcal{T}}_{AB}$ ,  $A, B \in \{\alpha, i\}$  are the adapted local components of a new stress-energy d-tensor  $\tilde{\mathcal{T}}$ . This new form of the Einstein equations will be treated detailed in the more general case of a *generalized metrical multi-time Lagrange space* [9].

ii) In the particular case of an autonomous metrical multi-time Lagrange space of electrodynamics (i. e.,  $g_{ij} = g_{ij}(x^k)$ ), the following Einstein equations of gravitational field

$$(E_1) \quad \begin{cases} H_{\alpha\beta} - \frac{H+r}{2}h_{\alpha\beta} = \mathcal{K}\mathcal{T}_{\alpha\beta} \\ r_{ij} - \frac{H+r}{2}g_{ij} = \mathcal{K}\mathcal{T}_{ij} \\ -\frac{H+r}{2}h^{\alpha\beta}g_{ij} = \mathcal{K}\mathcal{T}_{(i)(j)}^{(\alpha)(\beta)}, \end{cases}$$

$$(E_2) \quad \begin{cases} 0 = \mathcal{T}_{\alpha i}, & 0 = \mathcal{T}_{i\alpha}, & 0 = \mathcal{T}_{(i)\beta}^{(\alpha)} \\ 0 = \mathcal{T}_{\alpha(i)}^{(\beta)}, & 0 = \mathcal{T}_{i(j)}^{(\alpha)}, & 0 = \mathcal{T}_{(i)j}^{(\alpha)}, \end{cases}$$



hold good. It is remarkable that the new form  $(E'_1)$  of the Einstein equations of a metrical multi-time Lagrange space of electrodynamics reduces to the classical one, namely,

$$\begin{cases} H_{\alpha\beta} - \frac{H}{2}h_{\alpha\beta} = \mathcal{K}\tilde{T}_{\alpha\beta} \\ r_{ij} - \frac{r}{2}g_{ij} = \mathcal{K}\tilde{T}_{ij}. \end{cases}$$

iii) In order to have the compatibility of the Einstein equations, it is necessary that the certain adapted local components of the stress-energy d-tensor vanish "a priori".

From physical point of view, it is well known that the stress-energy d-tensor  $\mathcal{T}$  must verify the local *conservation laws*  $\mathcal{T}_{A|B}^B = 0$ ,  $\forall A \in \{\alpha, i, \binom{(\alpha)}{(i)}\}$ , where  $\mathcal{T}_A^B = G^{BD}\mathcal{T}_{DA}$ . Consequently, by a direct calculation, we find the following

**Theorem 4.4** *The conservation laws of the Einstein equations of a metrical multi-time Lagrange space  $ML_p^n$  are given by the formulas*

$$(4.4) \quad i) \text{ for } p = 1, \quad \begin{cases} \left[ \frac{R+S}{2} \right]_{/1} = R_{1|m}^m - P_{(1)1|m}^{(m)(1)} \\ \left[ R_j^m - \frac{R+S}{2}\delta_j^m \right]_{|m} = -P_{(1)j|m}^{(m)(1)} \\ \left[ S_{(1)(j)}^{(m)(1)} - \frac{R+S}{2}\delta_j^m \right]_{(m)}^{(1)} = -P_{(j)|m}^{m(1)}, \end{cases}$$

where  $R_1^i = g^{im}R_{m1}$ ,  $P_{(1)1}^{(i)} = h_{11}g^{im}P_{(m)1}^{(1)}$ ,  $R_j^i = g^{im}R_{mj}$ ,  $P_{(1)j}^{(i)} = h_{11}g^{im}P_{(m)j}^{(1)}$ ,  $P_{(j)}^{i(1)} = g^{im}P_{m(j)}^{(1)}$  and  $S_{(1)(j)}^{(i)(1)} = h_{11}g^{im}S_{(m)(j)}^{(1)(1)}$ ;

$$(4.5) \quad i) \text{ for } p \geq 2, \quad \begin{cases} \left[ H_\beta^\mu - \frac{H+R}{2}\delta_\beta^\mu \right]_{/\mu} = -R_{\beta|m}^m \\ \left[ R_j^m - \frac{H+R}{2}\delta_j^m \right]_{|m} = 0, \end{cases}$$

where  $H_\beta^\mu = h^{\mu\gamma}H_{\gamma\beta}$ ,  $R_j^i = g^{im}R_{mj}$  and  $R_\beta^i = g^{im}R_{m\beta}$ .

**Remarks 4.4 i)** In the case  $p > 2$ ,  $n > 2$ , taking into account the components  $\tilde{T}_{\alpha\beta}$  and  $\tilde{T}_{ij}$  of the new stress-energy d-tensor  $\mathcal{T}$  from  $(E'_1)$ , we point out that the conservation laws modify in the following simple and natural new form  $\tilde{T}_{\beta/\mu}^\mu = 0$ ,  $\tilde{T}_{j|m}^m = 0$  (see [9]).

ii) Considering an autonomous metrical multi-time Lagrange space of electrodynamics (i. e.,  $g_{ij} = g_{ij}(x^k)$ ), the conservation laws of the Einstein equations reduce

to

$$\left\{ \begin{array}{l} \left[ H_{\beta}^{\mu} - \frac{H+r}{2} \delta_{\beta}^{\mu} \right]_{/\mu} = 0 \\ \left[ r_j^m - \frac{H+r}{2} \delta_j^m \right]_{|m} = 0. \end{array} \right.$$

## 5 Conclusion

Note that all entities with geometrical or physical meaning from this paper was directly arised from the fundamental vertical metrical d-tensor  $G_{(i)(j)}^{(\alpha)(\beta)}$  of  $ML_p^n$ . This fact points out the *metrical character* and the naturalness of the metrical multi-time Lagrange theory of physical fields that we constructed. At the same time, the form of the invariance gauge group 1.9 of the fibre bundle of configurations,  $J^1(T, M) \rightarrow T \times M$ , allows us to appreciate the metrical multi-time Lagrangian field theory like a "*parametrized*" theory. In conclusion, the metrical multi-time Lagrangian theory of physical fields is, via the Marsden's classification of field theories [4], a "*metrical-parametrized*" one.

**Open problem.** The development of an analogous metrical multi-time Lagrangian geometry of physical fields on the jet space of order two  $J^2(T, M)$  is in our attention.

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